Fifth Annual Calculus Competition

February 19, 1994

1. Suppose that f and g are differentiable and $g(x) = xf(x^2 + 4)$. If $f(8) = 20$ and $f'(8) = -3$, what is $g'(2)$?

2. Let $f(x) = \cos 2x$. Find $f^{(1994)}(0)$, where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

3. Find $f'(\sqrt{\frac{\pi}{3}})$ if $f(x) = \displaystyle\int_0^{x^2} \sqrt{1 + 4\sin^2 t}\, dt$.

4. Sketch the graph of $f(x) = x^3 - 3x^2 - 9x$ labeling all relative extrema. For what values of b does the equation $f(x) = b$ have three distinct real solutions?

5. Suppose that f and f' are continuous on $[0, \ln 3]$ and that $f(0) = 0$, $f(\ln 3) = 4$, and $\int_0^{\ln 3} e^{2x} f'(x)dx = 10$. Find the value of $\int_0^{\ln 3} e^{2x} f(x)dx$.

6. Use integrals to find two positive numbers A and B such that $A < \displaystyle\sum_{n=1}^{100} \frac{1}{n^2} < B$.

7. Find a condition on A, B, and C that holds if and only if the series
$$\sum_{n=1}^{\infty} \left(\frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \right)$$
converges. Then give the sum of the series if this condition holds.

8. Let P and Q be points on the surfaces $2x - 2y + z = 5$ and $x^2 + y^2 + z^2 - 2x - 6y + 4z + 13 = 0$, respectively. Find the minimum distance between P and Q.

9. A rectangular box with sides parallel to the coordinate planes is contained in the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $4x + 2y + z = 8$. Find the largest volume such a box can have.

10. Evaluate $\displaystyle\int_0^{\sqrt{2\pi}} \int_{y/2}^{\sqrt{\pi/2}} \sin(x^2)\, dxdy$.