Third Annual Calculus Competition

February 15, 1992

1. Find the area of largest rectangle that can be inscribed in a semicircle with radius 4.

2. Evaluate the following if it exists: $\lim_{x\to\infty}\left(\ln x - \ln(\sqrt{x^2+1}+x)\right)$

3. Find the area of the region bounded by the curve $y = x^3 - 2x + 1$ and the line tangent to this curve at $(-1, 2)$.

4. Evaluate: $\displaystyle\int \frac{1}{1+e^x}\,dx$

5. Determine whether $\displaystyle\sum_{n=1}^{\infty}(-1)^n \cdot \frac{(2n)^n}{(2n)!}$ converges or diverges.

6. Let $f(t) = \dfrac{4}{1+t^2}$ and $G(x) = \displaystyle\int_0^x f(t)\,dt$.
   a) Find the power series for $f(t)$ about $t = 0$.
   b) Find the power series for $G(x)$ about $x = 0$.

7. Let $F(x) = \displaystyle\int_1^x \sqrt{t^2+2t}\,dt$.
   a) Find $F'(x)$.
   b) Find the length of the curve $y = F(x)$ for x in [1,2].

8. Let $f(x, y) = x^2e^{-2y}$, and let $P(2, 0)$ and $Q(-3, 1)$ be points.
   a) Determine the directional derivative of f at P in the direction from P to Q.
   b) Determine the unit vector that describes the direction from P that gives the most rapid increase in the function values of f.

9. Let P and Q be points on the surfaces $x^2 + y^2 + z^2 - 2x - 2y + 2z - 1 = 0$ and $x^2 + y^2 + z^2 + 2x + 4y - 10z + 29 = 0$, respectively. Find the minimum distance between P and Q.

10. Evaluate $\displaystyle\int_0^1\int_x^1 e^{y^2}\,dydx$