First Annual Calculus Competition

February 10, 1990

1. Suppose f and g are differentiable and $f(x) = x^2 g(x)$. If $g(2) = 3$ and $g'(2) = -2$, what is $f'(2)$?

2. Where is the function defined below (a) continuous; (b) differentiable?

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^2, & \text{if } 0 \leq x < 1; \\ 2x - 1, & \text{if } 1 \leq x < 2; \\ x^2 - 2x + 3, & \text{if } 2 \leq x. \end{cases}$$

3. Find the maximum of $f(x) = \frac{\ln x}{x}$ on the interval $(0, \infty)$.

4. Evaluate: $\displaystyle \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\pi}{n} \sin\left(\frac{k\pi}{n}\right)$

5. Evaluate: $\displaystyle \lim_{x \to \infty} \frac{\int_0^x \sqrt{1+t^2}\, dt}{x^2}$

6. Differentiate $f(x) = |\sec x|^x$

7. Suppose f is continuous and $f(-x) + f(x) = x^2$. Find $\displaystyle \int_{-1}^{1} f(x)\, dx$.

8. Suppose that

$$\sum_{n=2}^{\infty} \left(\frac{1}{c}\right)^n = 3.$$

   What is c?

9. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}.$$

   For what values of x is $f(x) > 0$?

10. Find the minimum value of $f(x, y) = xy^2$, if (x, y) must satisfy $x^2 + y^2 = 9$.

11. Evaluate: $\displaystyle \int_1^e \int_{\ln y}^{1} \frac{y}{1+e^x}\, dx dy$

12. Evaluate: $\displaystyle \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2}\, dy dx$