Twenty-Sixth Annual Calculus Competition

April 9, 2015

1. Suppose that $f(5) = 2$ and $f'(5) = -1$, and let $g(x) = x^3 f(x^2+1)$. Find $g'(2)$.

2. Evaluate: $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \tan^2\left(\frac{\pi k}{4n}\right)$

3. Find a positive value for m such that $e^x = mx$ has exactly one solution.

4. Let $f(x) = \dfrac{5x^2+18}{2x^2+7}$. Find the whole number nearest to $f(3^{20})$.

5. Find the area of the largest rectangle that can be inscribed in the curve $27x^2+y^4 = 108$.

6. Find the area inside the smaller loop of the graph of $r = 1 - 2\cos\theta$.

7. Let $s_1 = 1$ and $s_{n+1} = \dfrac{1}{s_1} + \dfrac{1}{s_2} + \cdots + \dfrac{1}{s_n}$ for each positive integer n. Determine whether the sequence $\{s_n\}$ converges, and if so, its limit.

8. Let $f(x) = x\sin(x^2)$. Find $f^{(2015)}(0)$.

9. Find all points (a, b, c) on the surface $z = x^2 - y^2 + 5$ such that the tangent plane at (a, b, c) passes through the points $(-4, 0, 0)$ and $(0, 2, 0)$.

10. Evaluate: $\displaystyle \int_0^9 \int_{\sqrt{x}}^3 \sqrt{1 - \frac{x}{y^2}}\, dy\, dx.$