Twenty-Fourth Annual Calculus Competition

April 11, 2013

1. Find all values c such that such that $y = x^2 - cx$ is tangent to the line $y = \frac{1}{2}x - 1$.

2. Let y be a function of x given implicitly by $x^2y + x^3 + xy^2 + y^3 = 1$. Find $\lim_{x\to\infty} \frac{y}{x}$. (You may assume that this limit exists.)

3. Show that $x^4 + 6x^2 \geq 16x - 9$ for all real numbers x.

4. Evaluate: $\lim_{x\to\infty} \left(\int_0^x u^8 e^{2u}\, du \right)^{1/x}$.

5. Find a value for C that will make the following function continuous on $(0, \infty)$.

$$f(x) = \begin{cases} \int_1^x \frac{1}{t}\, dt, & \text{if } 0 < x < 2 \\ C + \int_3^x \frac{2t}{1+t^2}\, dt, & \text{if } 2 \leq x. \end{cases}$$

6. On the polar curve $r = 3\sin\theta + 4\cos\theta$ find the maximum value of $y = r\sin\theta$.

7. Evaluate: $\displaystyle\int_0^\infty \frac{1}{(x^2+a^2)(a^2x^2+1)}\, dx$, where $a \neq 1$ is a positive number.

8. The Fibonacci numbers are defined as $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that the radius of convergence of

$$\sum_{n=0}^{\infty} a_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots$$

is at least $\frac{1}{2}$.

9. Find the volume of the solid in the first octant bounded by $x = 0$, $y = 0$, $z = 0$, and the plane tangent to $x^2yz^3 = 12$ at the point $(2, 3, 1)$.

10. Evaluate: $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} \sin(\pi\cos x)\, dx\, dy$.