Twenty-Third Annual Calculus Competition

April 12, 2012

1. Find all values c such that the line tangent to the graph of $f(x) = x^3 - 3x^2 - 5x + 12$ at $(c, f(c))$ is parallel to the line tangent to $y = e^{4x}$ at $(0, 1)$.

2. Consider a rectangle in the first quadrant with one vertex on the curve $y = \frac{100}{x^2+25}$ and two of its sides on the coordinate axes. Find the largest area that such a rectangle can have.

3. Evaluate: $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2k}{n(n+2k)}.$

4. Evaluate: $\displaystyle \lim_{x\to\infty} \frac{\int_0^x \tan^{-1}(t)\, dt}{\sqrt{4x^2+1}}.$

5. Evaluate: $\displaystyle \int_0^8 \frac{e^x}{e^x + e^{8-x}}\, dx.$

6. Suppose f'' is continuous, $f(0) = 1$, $f'(0) = -1$, and $f\left(\frac{\pi}{2}\right) = f'\left(\frac{\pi}{2}\right) = 2$. Find $\int_0^{\pi/2} (f(x) + f''(x)) \sin x \, dx$.

7. Let $f(x) = \dfrac{x^2}{1-2x^3}$. Find $f^{(2012)}(0)$.

8. A line with slope $m \neq 0$ and passing through $(0, 1)$ intersects the curve $x^2 + y^2 = 1$ at a point $(x, y) \neq (0, 1)$. Express x and y in terms of the parameter m.

9. Find all points (a, b, c) on the surface $z = x^2 + y^2 + 1$ such that the tangent plane at (a, b, c) passes through the points $(1, 0, 0)$ and $(0, 2, 0)$.

10. Evaluate: $\displaystyle \int_0^9 \int_{\sqrt{y}}^3 \frac{3y}{\sqrt{x^4 + 3y^2}}\, dx\, dy.$