Twenty-First Annual Calculus Competition

April 10, 2010

1. Find all points (x_0, y_0) on the curve $y = 4x - x^2$ such that the line tangent to this curve at (x_0, y_0) passes through $(1, 7)$.

2. A rectangular box with a square base and an open top is to have a volume of 48 cubic feet. If the sides cost \$2 per ft^2 and the base costs \$3 per ft^2, what are the dimensions of the cheapest such box that can be made?

3. Show that $x^6 \geq 6x - 5$ for all real numbers x.

4. Suppose that f is a function with continuous derivatives such that $f(0) = f(\frac{\pi}{2}) = 3$ and $\int_0^{\pi/2} f''(x)\sin(2x)\,dx = 8$. Find the value of $\int_0^{\pi/2} f(x)\sin(2x)\,dx$.

5. Evaluate: $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \tan\left(\frac{\pi i}{4n}\right).$$

6. Let f be a continuous function on $[1, \infty)$ satisfying $x\cos\pi x = \displaystyle\int_1^{x^2} f(t)\,dt$. Find $f(9)$.

7. Evaluate: $$\int_0^{\pi} \frac{e^x}{e^{\pi - x} + e^x}\,dx.$$

8. Determine the value of k so that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{k-1}\right)^n = 1.$

9. Let $f(x) = \dfrac{1}{1+x^3}$. Find the value of $f^{(2010)}(0)$.

10. Evaluate: $$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{x}{\sin x}\,dx\,dy.$$