Eighteenth Annual Calculus Competition

April 21, 2007

1. Find all points of intersection of the line tangent to $f(x) = x^4 + 5x^3 - 3x^2 - 6x - 1$ at $x = -1$ with the curve $y = f(x)$.

2. Evaluate: $\displaystyle\lim_{x\to\infty} \frac{e^{6x}}{\int_0^x \sqrt{t + e^{12t}}\, dt}$

3. Consider the triangle formed by the x- and y-axes and the line tangent to

$$y = f(x) = 1 + \frac{1}{x^2}$$

at the point $(a, f(a))$, where $a > 0$. For what value of a is the area of this triangle smallest?

4. Suppose that $f'(x) = \sqrt{9 + x^2}$ for all x and that $f(4) = 10$. Find the value of $\int_0^4 f(x)\, dx$.

5. Find the volume of the solid obtained by rotating about the x-axis the region bounded by $x + 2y = 6$, $x + 2y = 8$, the x-axis, and the y-axis.

6. Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

7. Suppose that y is a function of x that satisfies

$$\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{x^2}, \quad \text{and} \quad y = 0 \text{ at } x = \frac{2}{\pi}.$$

What is the value of y at $x = 3/\pi$?

8. Find the values of θ in $[0, 2\pi)$ for which the line tangent to the curve $r = \sqrt{2} + \sin\theta$ at (r, θ) is horizontal.

9. Find the (shortest) distance between the sphere $x^2 + y^2 + z^2 - 2x - 4z = 4$ and the line

$$\frac{x - 9}{8} = \frac{y - 3}{4} = z - 7.$$

10. Evaluate: $\displaystyle\int_0^1 \int_y^1 \frac{y}{x^6 + 1}\, dxdy$