Seventeenth Annual Calculus Competition

April 8, 2006

1. Find all points on the curve $x^2 + 2xy + 2y^2 = 45$ at which the slope of the tangent line is -2.

2. Let $f(x) = xe^{-x}$. Find $f^{(2006)}(0)$.

3. Assume that f'' is continuous and that $f(1) = 3$, $f'(1) = 2$, and $\int_0^1 f(x)\, dx = 5$. Find the value of $\int_0^1 x^2 f''(x)\, dx$.

4. Evaluate: $\lim\limits_{x \to \infty} \left(\sqrt[3]{x^3 + 18x^2} - x \right)$

5. Evaluate: $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i}{n^2} \sin\left(\dfrac{i^2 \pi}{n^2} \right)$.

6. Find the volume of the solid obtained by rotating about the x-axis the region bounded by $x^2 + y^2 = 2$, $x + y = 2$, and the x-axis.

7. Show that the equation $\cos x = x \sin x$ has exactly one solution in the interval $(0, \frac{\pi}{2})$.

8. Evaluate: $\displaystyle\int \dfrac{1}{e^{2x} + 3e^x + 2}\, dx$.

9. Find the maximum volume of a rectangular solid in the first octant with one vertex at the origin and the opposite vertex on the plane $2x + 3y + 6z = 18$.

10. Evaluate: $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 x^5 e^{x^2 y}\, dx\, dy$