Sixteenth Annual Calculus Competition

April 9, 2005

1. Find the volume of a pyramid with a square base and whose edges are all of length 1 meter.

2. Let $f(x) = \sin^4(x) + \cos^4(x)$. Find $f^{(100)}(x)$.

3. Evaluate: $\displaystyle \lim_{x \to 0} \frac{\sqrt[3]{1+3x} - 1 - x}{(1+x)^{50} - 1 - 50x}$.

4. Suppose that $f''(x) > 0$ for all x, and let a be a fixed real number. Show that $f(x) \geq f(a) + f'(a)(x-a)$ for all x.

5. Evaluate: $\displaystyle \int \frac{1}{\cos^2(x)\sin(x)}\, dx$.

6. Let
$$F(x) = \int_{-1}^{x} \sqrt{4+t^2}\, dt \quad \text{and} \quad G(x) = \int_{x}^{1} \sqrt{4+t^2}\, dt.$$
Find $(FG)'(0)$.

7. Determine whether the improper integral
$$\int_0^{\infty} (-1)^{\lfloor x^2 \rfloor}\, dx$$
converges or diverges, where $\lfloor \cdot \rfloor$ is the greatest integer function.

8. Show that if a and b are in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $|a-b| \leq |\tan a - \tan b|$.

9. Let P and Q be points on the surfaces $x - 2y + 2z = 15$ and $x^2 + y^2 + z^2 - 2x + 2y = 7$, respectively. Find the minimum distance between P and Q.

10. Evaluate: $\displaystyle \int_0^1 \int_y^1 \frac{y^3}{\sqrt{x^2+y^4}}\, dx dy$