Fifteenth Annual Calculus Competition

April 3, 2004

1. Find all points P on the curve $y = x^3 - 3x$ such that the line tangent to this curve at P passes through $(2, -2)$.

2. Suppose that f is differentiable on $(-\infty, \infty)$ and for all real x and h we have

$$f(x+h) - f(x) = hf'(x).$$

Show that $f(x) = ax + b$ for some constants a and b.

3. Let T be the triangle with vertices $(0,0)$, $(0,c^2)$, and (c,c^2), and let R be the region between $y = cx$ and $y = x^2$, where $c > 0$. Find

$$\lim_{c \to 0^+} A(T)/A(R),$$

where $A(\cdot)$ denotes area.

4. Evaluate: $\displaystyle \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{\sqrt{n^2 + 3kn}} \right)$

5. Find the values of a and b, where $a < b$, for which the integral

$$\int_a^b (24 - 2x - x^2)^{1/3}\, dx$$

has its largest value.

6. Let f be a positive, nonincreasing, differentiable function on $[0,1]$. Prove that

$$\int_0^1 f(x)(1 - 2x)\, dx \geq 0.$$

7. Evaluate: $\displaystyle \int_1^{16} \frac{1}{\sqrt{x} + \sqrt[4]{x}}\, dx.$

8. Let $x_1 = 1$ and $x_{n+1} = x_n + (1/x_n)$ for each positive integer n. Does $\{x_n\}$ converge? Prove your answer.

9. A rectangular box lies in the region in the first octant bounded by the coordinate planes and the surface $z = 16 - 4x^2 - y^2$ and has each of its sides parallel to one of the coordinate planes. Find the largest volume that this box can have.

10. Evaluate: $\displaystyle \int_0^4 \int_0^{\sqrt{y}} \frac{2y}{(4 + x^2)(1 + 12x - x^3)}\, dxdy$