Fourteenth Annual Calculus Competition

March 29, 2003

1. Two bicyclists Bob and Alice are travelling on straight roads that intersect perpendicularly at a point P. Bob is heading west toward P at 20 mph, and Alice is heading south toward P at 15 mph. At noon, Bob is 13 miles east of P and Alice is 16 miles north of P. Assuming that they continue at constant velocity, at what time are they closest to each other?

2. Find: $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{nx^{n-1}}{1+x}\,dx.$ (Hint: Integration by parts is a useful technique.)

3. Find the smallest possible constant A such that $\ln x \le Ax^2$ for all $x > 0$.

4. Find the volume of the solid generated by rotating about the x-axis the region bounded by $y = 1/\sqrt{x^2+1}$ and $y = 1/\sqrt{3x-1}$.

5. Let x and y be related by
$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt.$$
Show that $\frac{d^2y}{dx^2} = ky$ and find the constant k.

6. Evaluate: $\displaystyle\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right).$ (Hint: Consider pairing certain terms together.)

7. Find all values of x for which $\displaystyle\sum_{n=1}^{\infty}\frac{(2x-1)^n}{n}$ converges.

8. A line with slope m and passing through $(1,0)$ intersects the curve $x^2 - 3y^2 = 1$ at a point $(x,y) \neq (1,0)$. Express x and y in terms of the parameter m.

9. Suppose that f is a differentiable function such that $f_x(2,3) = 3$ and $f_y(2,3) = -4$. Let $g(x,y) = f(5x+3y, x^2-2y)$. In what direction from $(1,-1)$ does g increase most rapidly?

10. Evaluate: $\displaystyle\int_0^4\int_{y/2}^2 xy\sqrt{1+5x^4}\,dx\,dy$