Thirteenth Annual Calculus Competition

April 13, 2002

1. Find a cubic polynomial whose graph is tangent to $y = 1 - x$ at $(0, 1)$ and to $y = 2x - 3$ at $(2, 1)$.

2. Let $P(0, 3)$, $Q(0, 5)$, and $R(x, 0)$ be points. Determine a positive value for x such that $\angle PRQ$ is as large as possible.

3. Evaluate: $\displaystyle\lim_{x \to 1} \frac{\int_x^{x^2} e^{-t^2}\, dt}{x - 1}$

4. Show that
$$\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{\sqrt{1 + x}}\, dx \leq \frac{1}{3}$$

5. Evaluate: $\displaystyle\int \ln(1 + x^2)\, dx.$

6. Find the area of the region that lies outside $r = 2$ and inside $r = 4 \sin 5\theta$.

7. Find the value of
$$\left\lfloor \sum_{n=1}^{400} \frac{1}{\sqrt{n}} \right\rfloor,$$
where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (Hint: Integration may be useful.)

8. Find the (shortest) distance between the lines
$$\frac{x - 1}{2} = \frac{y}{3} = \frac{z}{2} \qquad \text{and} \qquad \frac{x - 3}{2} = \frac{y}{5} = \frac{z - 2}{4}$$

9. Find the maximum and minimum values of $f(x, y) = 7x^2 + 4xy + 3y^2$ subject to $x^2 + y^2 = 1$.

10. Evaluate: $\displaystyle\int_0^1 \int_0^{\arccos y} \cos(\sin x)\, dx dy$